PRINCIPLES OF ANALYSIS
TOPIC VI: OPEN AND CLOSED SETS

PAUL L. BAILEY

Abstract. We define open sets, closed sets, and accumulation points, and prove a third version of the Bolzano-Weierstrass Theorem wrapped in this terminology. We define connected and compact sets, and prove the Heine-Borel Theorem.

1. Open Sets

Definition 1. A subset $U \subset \mathbb{R}$ is called *open* if

$$\forall u \in U \; \exists \epsilon > 0 \; \ni \; |x - u| < \epsilon \Rightarrow x \in U.$$

This definition can be restated in terms of neighborhoods.

Definition 2. Let $x \in \mathbb{R}$. An *ϵ-neighborhood* of x is an open interval of the form $(x - \epsilon, x + \epsilon)$, where $\epsilon > 0$.

More generally, a *neighborhood* of x is a subset $Q \subset \mathbb{R}$ such that there exists $\epsilon > 0$ with $(x - \epsilon, x + \epsilon) \subset Q$.

So, a set $U \subset \mathbb{R}$ is open if every point in U is surrounded by an ϵ-neighborhood which is completely contained in U.

If $\mathcal{C}$ is a collection of subsets of a given set X, then the *union* and *intersection* of $\mathcal{C}$ are

$$\cup\mathcal{C} = \{x \in X \mid x \in C \text{ for some } C \in \mathcal{C}\};$$
$$\cap\mathcal{C} = \{x \in X \mid x \in C \text{ for all } C \in \mathcal{C}\}.$$

Proposition 1. *Let $\mathcal{T}$ denote the collection of all open subsets of $\mathbb{R}$. Then*

(a) $\varnothing \in \mathcal{T}$ *and* $\mathbb{R} \in \mathcal{T}$*;*
(b) *if* $\mathcal{O} \subset \mathcal{T}$*, then* $\cup\mathcal{O} \in \mathcal{T}$*;*
(c) *if* $\mathcal{O} \subset \mathcal{T}$ *is finite, then* $\cap\mathcal{O} \in \mathcal{T}$.

Proof.

(a) The condition for openness is vacuously satisfied by the empty set. For $\mathbb{R}$, consider $x \in \mathbb{R}$. Then $(x - 1, x + 1) \subset \mathbb{R}$. Thus $\mathbb{R}$ is open.

(b) Let $\mathcal{O} \subset \mathcal{T}$; that is, $\mathcal{O}$ is a collection of open sets. Select $x \in \cup\mathcal{O}$. Then $x \in U$ for some $U \in \mathcal{O}$. Since U is open, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Since $U \subset \cup\mathcal{O}$, it follows that $(x - \epsilon, x + \epsilon) \subset \cup\mathcal{O}$. Thus $\cup\mathcal{O}$ is open.

(c) Let $\mathcal{O} \subset \mathcal{T}$ be a finite collection of open sets. Since $\mathcal{O}$ is finite, we may write $\mathcal{O} = \{U_1, U_2, \dots, U_n\}$, where U_i is an open set for $i = 1, \dots, n$. If $\cap\mathcal{O}$ is empty, we are done, so assume that it nonempty, and select $x \in \cap\mathcal{O}$. For each i, there exists ϵ_i such that $(x - \epsilon_i, x + \epsilon_i) \subset U_i$. Set $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then $(x - \epsilon, x + \epsilon) \subset \cap\mathcal{O}$. Thus $\cap\mathcal{O}$ is open. □

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Proposition 2. *Let $\mathcal{O}$ be a collection of open intervals. If $\cap\mathcal{O}$ is nonempty, then $\cup\mathcal{O}$ is an open interval.*

Proof. By hypothesis, there exists $x \in \cap\mathcal{O}$. Write $\mathcal{O}$ as a family of sets:

$$\mathcal{O} = \{O_\alpha \mid \alpha \in A\},$$

where A is an indexing set. Now O_α is an open interval; we label its endpoints by letting $O_\alpha = (a_\alpha, b_\alpha)$, where $a_\alpha, b_\alpha \in \mathbb{R} \cup \{\pm\infty\}$. Set

$$a = \inf\{a_\alpha \mid \alpha \in A\} \quad \text{and} \quad b = \sup\{b_\alpha \mid \alpha \in A\}.$$

Claim: $\cup\mathcal{O} = (a, b)$. We prove both directions of containment.

($\subset$) Let $y \in \cup\mathcal{O}$. Then $y \in O_\alpha$ for some α. Thus $a \leq a_\alpha < y < b_\alpha \leq b$, so $y \in (a, b)$.

($\supset$) Let $y \in (a, b)$. Assume that $y \leq x$; the proof for $y \geq x$ is analogous. Now $a < y$, and since $a = \inf\{a_\alpha \mid \alpha \in A\}$, so there exists $\alpha \in A$ such that $a \leq a_\alpha < y$. Also $x \in O_\alpha$ so $a_\alpha < y \leq x < b_\alpha$; thus $y \in (a_\alpha, b_\alpha) = O_\alpha$, and $y \in \cup\mathcal{O}$. □

Proposition 3. *Let $U \subset \mathbb{R}$. Then U is open if and only if there exists a collection $\mathcal{O}$ of disjoint open intervals such that $U = \cup\mathcal{O}$.*

Proof. Let $a \in U$, and set $\mathcal{O}_a = \{O \subset U \mid O \text{ is an open interval and } a \in O\}$. Set $O_a = \cup\mathcal{O}_a$. By the previous proposition, O_a is an open interval.

Now suppose that $a, b \in U$ and suppose that $O_a \cap O_b \neq \varnothing$. Then there exists $c \in O_a \cap O_b$, so $O = O_a \cup O_b$ is an open interval by the Proposition 2. Also $a \in O$, so $O \in \mathcal{O}_a$, so $O \subset O_a$. Similarly, $O \subset O_b$. This shows that $O_a = O_b$.

Let $\mathcal{O} = \{O_a \mid a \in U\}$. This is a collection of disjoint open intervals contained in U, and every element of U is in one of these open intervals, so $U = \cup\mathcal{O}$. □

2. Closed Sets

Definition 3. A subset $F \subset \mathbb{R}$ is *closed* if its complement $\mathbb{R} \setminus F$ is open.

We may characterize the collection $\mathcal{F}$ of closed subsets of $\mathbb{R}$ in a manner analogous to our characterization of $\mathcal{T}$, the collect of open subsets of $\mathbb{R}$, by the use of *DeMorgan's Laws.*

Proposition 4. (DeMorgan's Laws)
Let X be a set and let $\{A_\alpha \mid \alpha \in I\}$ be a family of subsets of X. Then

$$\bigcap_{\alpha\in I}(X \setminus A_\alpha) = X \setminus \Big(\bigcup_{\alpha\in I} A_\alpha\Big);$$

$$\bigcup_{\alpha\in I}(X \setminus A_\alpha) = X \setminus \Big(\bigcap_{\alpha\in I} A_\alpha\Big).$$

Proposition 5. *Let $\mathcal{F}$ denote the collection of all closed subsets of $\mathbb{R}$.*

(a) *$\varnothing \in \mathcal{F}$ and $\mathbb{R} \in \mathcal{F}$;*
(b) *if $\mathcal{C} \subset \mathcal{F}$, then $\cap\mathcal{C} \in \mathcal{F}$;*
(c) *if $\mathcal{C} \subset \mathcal{F}$ is finite, then $\cup\mathcal{C} \in \mathcal{T}$.*

Proof. Apply DeMorgan's Laws to Proposition 1. □

Proposition 6. *Let $F \subset \mathbb{R}$. Then F is closed if and only if every sequence in F which converges in $\mathbb{R}$ has a limit in F.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that F is closed, and let (a_n) be a sequence in F which converges to $a \in \mathbb{R}$. We wish to show that $p \in F$. Suppose not; then $p \in \mathbb{R} \setminus F$. This set is open, so there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset \mathbb{R} \setminus F$. Thus there exists $N \in \mathbb{N}$ such that $a_n \in \mathbb{R} \setminus F$ for all $n \geq N$. This contradicts that the sequence is in F.

($\Leftarrow$) Suppose that F is not closed; we wish to construct a sequence in F which converges to a point not in F. Since F is not closed, then $\mathbb{R} \setminus F$ is not open. This means that there exists a point $x \in \mathbb{R} \setminus F$ such that for every $\epsilon > 0$, $(x - \epsilon, x + \epsilon)$ is not a subset of $\mathbb{R} \setminus F$; that is, $(x - \epsilon, x + \epsilon)$ contains a point in F. For $n \in \mathbb{N}$, let $x_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap F$. Then (x_n) is a sequence in F, but $\lim_{n \to \infty} x_n = x \notin F$. ☐

3. Accumulation Points

Definition 4. Let $S \subset \mathbb{R}$ and let $x \in \mathbb{R}$. We say that x is an *accumulation point* of S if for every $\epsilon > 0$ there exists $s \in S$ such that $0 < |s - x| < \epsilon$.

This definition may be restated in terms of deleted neighborhoods.

Definition 5. A *deleted neighborhood* of $x \in \mathbb{R}$ is a set of the form $Q \setminus \{x\}$, where Q is a neighborhood of x.

Thus x is an accumulation point of S if every deleted neighborhood of x contains an element of S. We note that an accumulation point of a set S may or may not be an element of S.

Proposition 7. *Let $F \subset \mathbb{R}$. Then F is closed if and only if F contains all of its accumulation points.*

Proof. Prove both directions.

($\Rightarrow$) Suppose F is closed, and let $x \in \mathbb{R}$. Suppose $x \notin F$; we show that x is not an accumulation point of F. Since $x \in F$, then $x \in \mathbb{R} \setminus F$, which is open. Therefore there exists $\epsilon > 0$ such that $U = (x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus F$. Then $U \setminus \{x\}$ is a deleted neighborhood of x whose intersection with F is empty, and x is not an accumulation point of F.

($\Leftarrow$) Suppose F contains all of its accumulation points. We show that the complement of F is open. Let $x \in \mathbb{R} \setminus F$. Then x is not an accumulation point of F. Then there exists a deleted neighborhood U of x such that $U \subset \mathbb{R} \setminus F$. This neighborhood contains a deleted epsilon neighborhood, say $(x - \epsilon, x + \epsilon) \setminus \{x\}$. This set is in the complement of F, and since $x \notin F$, we have $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus F$. Thus $\mathbb{R} \setminus F$ is open, so F is closed. ☐

Theorem 1. (Bolzano-Weierstrass Theorem Version III)
Every bounded infinite set of real numbers has an accumulation point.

Proof. Let S be a bounded infinite set. Since S is infinite, there exists an injective function $s : \mathbb{N} \to S$; view this as a sequence (s_n). This sequence is bounded, and consequently has a convergent subsequence; say (s_{n_k}) converges to $q \in \mathbb{R}$. Suppose that $q = s_{n_M}$ for some $M \in \mathbb{N}$; then, since (s_n) is injective, $n > M \Rightarrow s_n \neq s$. If no such M exists, let $M = 0$.

We claim that q is an accumulation point of S. Let $\epsilon > 0$; to show that q is an accumulation point of S, we need to find an element of $S \smallsetminus \{q\}$ within ϵ of q. But since (s_{n_k}) converges to q, there exists $N \in \mathbb{N}$ such that $k \geq N \Rightarrow |s_{n_k} - q| < \epsilon$. Let $K = \max\{M, N\} + 1$ and let $s = s_{n_K}$. Then $s \in S$, $s \neq q$, and $|s - q| < \epsilon$. Thus q is an accumulation point of S. □

4. Connected Sets

Definition 6. A subset $A \subset \mathbb{R}$ is *disconnected* if there exist disjoint open sets $U_1, U_2 \subset \mathbb{R}$ with $A \cap U_1 \neq \varnothing$ and $A \cap U_2 \neq \varnothing$ such that $A \subset (U_1 \cup U_2)$. Otherwise, we say that A is *connected*.

Intuitively, we imagine that the connected sets are intervals. Before we show this, let us characterize intervals in a a variety of ways.

Proposition 8. *Let $A \subset \mathbb{R}$ contain at least two elements. The following conditions on A are equivalent:*

- **(a)** *$a_1, a_2 \in A$ and $a_1 < a_2$ implies $[a_1, a_2] \subset A$;*
- **(b)** *$(\inf A, \sup A) \subset A$;*
- **(c)** *A is an interval.*

Proof. We show that **(a)** $\Rightarrow$ **(b)** $\Rightarrow$ **(c)** $\Rightarrow$ **(a)**. We will assume that A is bounded; minor adjustments will handle the cases where $\inf A = -\infty$ or $\sup A = \infty$.

(a) $\Rightarrow$ **(b)** Suppose that for every $a_1, a_2 \in A$ with $a_1 < a_2$, we have $[a_1, a_2] \subset A$. Let $c \in (\inf A, \sup A)$, so that $\inf A < c < \sup A$. Then there exists $a_1 \in A$ such that $\inf A \leq a_1 < x$; also, there exists $a_2 \in A$ such that $c < a_2 < \sup A$. Now $c \in [a_1, a_2]$ and $[a_1, a_2] \subset A$; thus $c \in A$.

(b) $\Rightarrow$ **(c)** Suppose that $(\inf A, \sup A) \subset A$. By definition of supremum and infimum, this implies that $A \smallsetminus (\inf A, \sup A) \subset \{\inf A, \sup A\}$.

We wish to show that A is an interval. But a bounded interval is a set of the form $\{x \in \mathbb{R} \mid a < x < b\} \cup E$, where $a, b \in \mathbb{R}$, $a < b$, and E is a subset of $\{a, b\}$. If we let $a = \inf A$ and $b = \sup A$, we have $A = (a, b) \cup E$, where $E = A \smallsetminus (a, b) \subset \{a, b\}$.

(c) $\Rightarrow$ **(a)** Suppose that A is a bounded interval; then there exist $a, b \in \mathbb{R}$ with $a < b$ and $A = \{x \in \mathbb{R} \mid a < x < b\} \cup E$, where $E \subset \{a, b\}$. Let $a_1, a_2 \in A$ with $a_1 < a_2$, and let $c \in [a_1, a_2]$.

We wish to show that $c \in A$. If $c = a_1$ or $c = a_2$, we are done, so assume $c \neq a_1$ and $c \neq a_2$. Then $a \leq a_1 < c < a_2 \leq b$, so $c \in \{x \in \mathbb{R} \mid a < c < b\} \subset A$. □

Proposition 9. *Let $A \subset \mathbb{R}$ contain at least two elements. Then A is connected if and only if A is an interval.*

Proof. We prove both directions of the implication.

($\Rightarrow$) Suppose that A is not an interval. Then there exist $a_1, a_2 \in A$ with $a_1 < a_2$ such that $[a_1, a_2]$ is not contained in A, so there exists $c \in [a_1, a_2]$ such that $c \notin A$. Set $U_1 = (-\infty, c)$ and $U_2 = (c, \infty)$; then $a_1 \in U_1$, $a_2 \in U_2$, and $A \subset U_1 \cup U_2$. Thus A is disconnected.

($\Leftarrow$) Suppose that A is an interval. Then for every $a_1, a_2 \in A$ with $a_1 < a_2$, we have $[a_1, a_2] \subset A$.

Let U_1 and U_2 be open sets with $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, and $A \subset U_1 \cup U_2$. We wish to show that $U_1 \cap U_2 \neq \varnothing$.

Let $a_1 \in U_1$ and $a_2 \in U_2$; without loss of generality, assume that $a_1 < a_2$. Let $c = \sup U_1 \cap [a_1, a_2]$. Clearly $c \in [a_1, a_2]$, so either $c \in U_1$ or $c \in U_2$.

Case 1: $c \in U_1$

Since U_1 is open, there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U_1$. Let $d = \min\{\frac{\epsilon}{2}, \frac{a_2 - c}{2}\}$; then $c + d$ is also in U_1 and in $[a_1, a_2]$. By definition of c, we must have $d = 0$, which implies that $\frac{c - a_2}{2} = 0$, which implies that $c = a_2$. Since $a_2 \in U_2$, we have $c \in U_1 \cap U_2$.

Case 2: $c \in U_2$

Since U_2 is open, there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset U_2$. But by the definition of c, there exists $b \in U_1 \cap [a_1, a_2]$ such that $b \in (c - \epsilon, c) \subset U_2$, so $b \in U_1 \cap U_2$. $\square$

5. Compact Sets

Let $A \subset \mathbb{R}$. A *cover* of A is a collection $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ of subsets of $\mathbb{R}$ such that $A \subset \cup\mathcal{C}$.

Let $\mathcal{C}$ be a cover of $A \subset \mathbb{R}$. We say that $\mathcal{C}$ is an *open cover* if every member $U \in \mathcal{C}$ is an open subset of $\mathbb{R}$. We say that $\mathcal{C}$ is a *finite cover* if $\mathcal{C}$ is a finite set.

Note that the modifier *open* refers to the sets inside $\mathcal{C}$, whereas the modifier *finite* refers to the collection $\mathcal{C}$ itself.

A *subcover* of $\mathcal{C}$ is a subset $\mathcal{D} \subset \mathcal{C}$ such that $A \subset \cup\mathcal{D}$.

We say that A is *compact* if every open cover of A has a finite subcover.

Example 1. Let $A = \mathbb{Z}$. Let $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{Z}\}$. Then $\mathcal{C}$ is an open cover of $\mathbb{Z}$ with no finite subcover. Thus $\mathbb{Z}$ is not compact.

Example 2. Let $A = (0,1)$. Let $I_n = (0, 1 - \frac{1}{n})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{N}\}$. Then $\mathcal{C}$ is an open cover of $(0,1)$ with no finite subcover. Thus $(0,1)$ is not compact.

Proposition 10. *Let $A = \{a_1, \dots, a_n\}$ be a finite set. Then A is compact.*

Proof. Let $\mathcal{C}$ be an open cover of A. Then for each $a_i \in A$, there exists and open set $U_i \in \mathcal{C}$ such that $a_i \in U_i$. Then $A \subset \cup_{i=1}^n U_i$, and $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{C}$. Thus A is compact. □

Proposition 11. *Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed interval $[a,b] \subset \mathbb{R}$ is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $[a,b]$.

Let $x \in [a,b]$ and let $U_x \in \mathcal{C}$ be an open set which contains x. Then there exists $\epsilon_x > 0$ such that $(x - \epsilon_x, x + \epsilon_x) \subset U_x$. Let

$$B = \{x \in [a,b] \mid [a,x] \text{ can be covered by a finite subcover of } \mathcal{C}\}.$$

Note that B is nonempty, since the closed interval $[a, a + \frac{\epsilon_a}{2}] \subset U_a$, and $\{U_a\}$ is a finite subcover of $\mathcal{C}$, so for example $a + \frac{\epsilon_a}{2} \in B$.

Let $z = \sup B$; clearly $a + \frac{\epsilon_a}{2} \leq z \leq b$. We claim that $z \in B$, and that $z = b$. To see this, let $\epsilon = \min\{\epsilon_z, z - a\}$. Then $z - \frac{\epsilon}{2} \in B$. Let $\mathcal{D}$ be a finite subcover of $\mathcal{C}$ which covers $[a, z - \frac{\epsilon}{2}]$. Then $\mathcal{D} \cup \{U_z\}$ covers $[a,z]$, so $z \in B$. Now suppose that $z < b$, and set $\delta = \min\{\epsilon, z - b\}$. Then $z < z + \frac{\delta}{2} < b$, and $\mathcal{D} \cup \{U_z\}$ covers $[a, z + \frac{\delta}{2}]$; since $z + \frac{\delta}{2} \in [a,b]$, this contradicts the definition of z. Thus $z = b$. This completes the proof. □

Proposition 12. *Let $A \subset \mathbb{R}$ be compact and let $F \subset A$ be closed. Then F is compact.*

Proof. Let $\mathcal{C}$ be an open cover of F. Let $U = \mathbb{R} \smallsetminus F$; since F is closed, U is open. Let $\mathcal{B} = \mathcal{C} \cup \{U\}$. Now $\mathcal{B}$ is an open cover of A. Since A is compact, let $\mathcal{U}$ be a finite subcover of A. Since $F \subset A$, then $\mathcal{U}$ is also a finite open cover of F. Let $\mathcal{V} = \mathcal{U} \smallsetminus \{U\}$; now $\mathcal{V}$ is still a finite open cover of F, and $\mathcal{V}$ is a subcover of $\mathcal{C}$. Thus F is compact. □

Theorem 2. (Heine-Borel Theorem)
Let $A \subset \mathbb{R}$. Then A is compact if and only if A is closed and bounded.

Proof. We prove both directions.

($\Rightarrow$) Suppose that A is compact; we wish to show that A is closed and bounded.

Cover A with sets of the form $(-n, n)$, for $n \in \mathbb{N}$. Since A is compact, there exists a finite subcover. This subcover contains an interval of maximum length, say $(-M, M)$, and clearly $A \subset (-M, M)$. Thus $A \subset [-M, M]$, and A is bounded.

To show that A is closed, we show that its complement is open. Let $B = \mathbb{R} \setminus A$. Let $b \in B$. For each point $a \in A$, set $\epsilon_a = |b - a|/2$, $I_a = (a - \epsilon, a + \epsilon)$, and $J_a = (b - \epsilon, b + \epsilon)$. Let $\mathcal{I} = \{I_a \mid a \in A\}$. Then $\mathcal{I}$ is an open cover of A, and so it has a finite subcover $\{I_{a_1}, \dots, I_{a_n}\}$. The open set $\cup_{i=1}^n I_{a_i}$ contains A and is disjoint from the set $\cap_{i=1}^n J_{a_i}$, which is also open and contains b. Thus B is open.

($\Leftarrow$) Suppose that A is closed and bounded; we wish to show that A is compact. Since A is bounded, there exists $M > 0$ such that $A \subset [-M, M]$. The set $[-M, M]$ is a closed interval, and so it is compact by Proposition 11. Thus A is a closed subset of a compact set, and therefore is compact by Proposition 12. $\square$

Proposition 13. *Let K be a compact set. Then* $\inf K \in K$ *and* $\sup K \in K$.

Proof. Since K is bounded, then $\sup K$ exists as a real number, say $b = \sup K$. Suppose $b \notin K$; then $\{(-\infty, b - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open cover of K with no finite subcover, contradicting that K is compact. Thus $b \in K$. Similarly, $\inf K \in K$. $\square$

6. Problems

Problem 1. Let (a_n) be a bounded sequence in $\mathbb{R}$ and let

$$\Lambda = \{q \in \mathbb{R} \mid q \text{ is a cluster point of } (a_n)\}.$$

(a) Show that Λ is closed.
(b) Show that Λ is bounded.
(c) Show that Λ is compact.

Department of Mathematics and CSci, Southern Arkansas University
E-mail address: plbailey@saumag.edu